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# The Energy of a Vibrating String

We have derived (in two different ways) the general expression describing a string, pinned at the ends, vibrating in normal mode #*n*: $y(x,t) = f(x)\cos(\omega_n t - \phi_n)$ or, explicitly writing the function $f(x)$:

$$y(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t - \phi_n)$$

**What is the energy of the vibrating string?** We know that in general it has kinetic energy, since it's moving, and potential energy, since the string is tense and deformed. The total energy $E = KE + PE$ is constant. (We're ignoring damping. With damping, we know that $E(t) = E_0 e^{-\gamma t}$.) Since *E* is constant, we can make our lives easier by picking a "simple" configuration in which to evaluate *E*. We realize that when the string is straight (i.e. $y = 0$)

- the kinetic energy is nonzero, since the string is moving past this "flat" configuration and
- the potential energy is zero, since the string is not deformed.

Therefore at this particular instant in time, $E = KE$, and so it suffices to calculate the kinetic energy. What is this instance in time? It's determined by $\omega_n t - \phi_n = \pi/2$, since the cosine factor above is zero then.

The kinetic energy for a segment of string located between $x$ and $x + dx$ is just the usual "$\frac{1}{2}mv^2$" for this segment, which is $\frac{1}{2}\mu dx(\dot{y})^2$, where $\mu$ is the mass density. From the above expression:

$$\dot{y}(x,t) = -\omega_n A_n \sin\left(\frac{n\pi}{L}x\right)\sin(\omega_n t - \phi_n).$$

At our "special" point in time, $\dot{y}(x,t^*) = -\omega_n A_n \sin\left(\frac{n\pi}{L}x\right)$, the kinetic energy of our small piece of string is $\frac{1}{2}\mu dx(\dot{y})^2 = \frac{1}{2}\mu dx\left(\omega_n A_n \sin(n\pi x/L)\right)^2$. The total kinetic energy is simply

$$KE = \int_0^L "dKE" = \int_0^L \frac{1}{2}\mu\omega_n^2 A_n^2 \sin^2\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}\mu\omega_n^2 A_n^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx$$

which is a standard integral you've seen recently, so:

$$KE = \frac{1}{2}\mu\omega_n^2 A_n^2 \frac{L}{n\pi}\left[\frac{n\pi}{2} - \frac{\sin(2n\pi)}{4}\right] = \frac{1}{4}\mu\omega_n^2 A_n^2 L$$

Therefore the total energy **at all times** is $\boxed{E = \frac{1}{4}\mu\omega_n^2 A_n^2 L}$.

Inserting our expression for the normal mode frequencies, $\omega_n = \frac{n\pi}{L}\sqrt{\frac{\tau}{\mu}}$, yields: $\boxed{E = \frac{\pi^2}{4}\frac{A_n^2\tau}{L}n^2}$.

**Note that** $E \sim n^2$. For a given amplitude, more energy is required to excite higher modes!